

Centrifugal force reversal from the perspective of rigidly rotating observer

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Abstract

In previous studies the dynamics of the relativistic particle moving along the rotating pipe was investigated. The simple gedanken experiment was considered. It was shown that at large enough velocities a centrifugal force acting on the bead changes its usual sign and attracts towards the rotation axis. The authors studied motion of the particle along the rotating straight pipe in the frame of the observer located in the center of rotation, also dynamics of centrifugally accelerated relativistic particle was studied in the laboratory frame. In the both cases it was shown that centrifugal force changes sign. Recently the problem was studied in the frame of stationary observers. It was argued that centrifugal acceleration reversal is not frame invariant effect. In the present paper we consider motion of particle along the rotating straight line in the frame of an arbitrary stationary observer located at certain distance from the center of rotation and rigidly rotating with constant angular velocity. It is shown that any stationary observer could detect reversal of centrifugal acceleration.

I. INTRODUCTION

In [1] the authors studied the two-dimensional, relativistic motion of a particle along a rotating field line. In this study, the case of a straight trajectory was considered. The model was highly idealized and purely mechanical but it had the advantage of having exact analytic solutions. Authors found that the particle does not cross the “light cylinder” radius R_c (the light cylinder radius is defined as a hypothetical surface where the linear velocity of rotation equals the speed of light, $R_c \equiv c/\omega$, where ω is the angular velocity of rotation and c denotes the speed of light). In this study it was also shown that if the initial radial velocity of the particle $v_{r0} \geq c/\sqrt{2}$, then the radial acceleration is initially negative (cf. the special-relativistic effect of the “centrifugal force reversal”, [1]). In other words, the motion of a particle moving along a rotating, straight linear “pipe” is limited by R_c , and during the course of its motion the particle may experience not only acceleration, but also deceleration. Behavior of centrifugal force in the frame work of the general relativity was studied in [2] and [3], it was found that centrifugal force could change sign. The main idea of the ‘pipe-bead’ (‘rotator-pipe-bead’) *gedanken* (see [4], [1], [5], [6], [7]) experiments was to mimic the common situation in relativistic and rotating astrophysical flows, where the plasma particles are forced to move along the field lines of governing magnetic fields.

Recently in [8] was argued that the effect discussed by [1] is not frame-invariant and disappears if one uses frame-invariant quantities. Thus, in special relativity, there is no reversal of centrifugal acceleration. The effect seen in [1] is a time dilation ([9]) and it describes an unphysical coordinate acceleration.

In [8] author raised interesting idea to study the centrifugal force reversal in the frame of different observers. In the present study the motion of the particle is considered in the of the observer located at certain distance from the center of rotation and rigidly rotating with constant angular velocity.

II. MAIN CONSIDERATION

As in [1]) and [8] the problem is considered in geometrical units in which $G = c = 1$, while the metric signature is: $(-1, 1, 1)$.

Let us first consider the the dynamics of the particle moving along the rotating wire in

the frame of reference rotating with the pipe-bead system (i.e. rotating frame - RF). In order to do this, we first need to switch to the frame, rotating with the angular velocity ω . Employing the transformation of variables:

$$T = t, \quad (1)$$

$$\tilde{\varphi} = \varphi + \omega t \quad (2)$$

one arrives to the metric:

$$ds^2 = -(1 - \omega^2 r^2)dt^2 + 2\omega^2 dt d\varphi + r^2 d\varphi^2 + dr^2. \quad (3)$$

For the straight pipe ($\varphi = \varphi_0 = \text{const}$) case, Eq. 3 reduces to the metric $ds^2 = -(1 - \omega^2 r^2)dt^2 + dr^2$, which is the basic metric for the [1]) study.

The resulting metric tensor

$$\|g_{\alpha\beta}\| = \begin{pmatrix} -(1 - \omega^2 r^2) & 0 \\ 0 & 1 \end{pmatrix}, \quad (4a)$$

we find:

$$\Delta \equiv [-\det(g_{\alpha\beta})]^{1/2} = (1 - \omega^2 r^2)^{1/2}, \quad (4b)$$

For this relatively simple, but non-diagonal, two-dimensional space-time one can develop the “1 + 1” formalism. In doing so we can follow, as a blueprint, the well-known “3 + 1” formalism, widely used in the physics of black holes ([10], [11], [12]). Namely, we first introduce the definitions of the *lapse function*:

$$\alpha \equiv \frac{\Delta}{g_{rr}} = \sqrt{1 - \omega^2 r^2}, \quad (5)$$

Within this formalism Eq. 3 can be presented in the following way:

$$ds^2 = -\alpha^2 dt^2 + g_{rr} dr^2. \quad (6)$$

Note that for the metric tensor, Eq. 4a, t is the cyclic coordinate and, moreover, in the RF the motion of the bead inside the pipe is geodesic, i.e. there are no external forces acting on it. Hence the proper energy of the bead, E_0 , must be a conserved quantity. Employing the definition of the four velocity $U^\alpha \equiv dx^\alpha/d\tau$ one can write:

$$E_0 \equiv -U_t = -U^t g_{tt} = \text{const}. \quad (7)$$

On the other hand, the basic four-velocity normalization condition $g_{\alpha\beta}U^\alpha U^\beta = -1$ requires:

$$U^t = [-g_{tt} - g_{rr}v_r^2]^{-1/2}. \quad (8a)$$

This equation, written explicitly, has the following form:

$$U^t = [1 - \omega^2 r^2 - v_r^2]^{-1/2}. \quad (8b)$$

Recalling the definition of the Lorentz factor $\gamma(t) = \frac{1}{\sqrt{1 - r^2\omega^2 - v_r^2}}$, with $v_r = dr/dt$, one can easily see that:

$$U^t = [1 - r^2\omega^2 - v_r^2]^{-1/2} = \gamma(t). \quad (8c)$$

It is important to note that the conserved proper energy of the bead, E_0 , defined by Eq. 7, may be written simply as:

$$E_0 = \gamma(t)[1 - r(t)^2\omega^2] = \text{const.} \quad (9)$$

From Eq. 9 follows that radial velocity of particle is:

$$(\partial_t r)^2 = (1 - \omega^2 r^2) \left(1 - \frac{1 - \omega^2 r^2}{E_0^2} \right). \quad (10)$$

Now let us consider motion of the test particle in the frame of the observer, located at distance R_{SO} from the center of rotation and moving with velocity ωR_{SO} perpendicular to $\overrightarrow{R_{SO}}$. Let us consider the case when azimuthal coordinate of the observer coincides with azimuthal coordinate of the test particle. The above introduced frame instantaneously is equivalent to the inertial frame moving with velocity ωR_{SO} perpendicular to $\overrightarrow{R_{SO}}$ at the point of observer location. Eq. 11 gives relation between time interval (dt) measured by observer located at the center of rotation and rotating with angular velocity ω and time interval (dt_{SO}) measured by stationary observer located at R_{SO} and rotating with angular velocity ω . Eq. 12 relates change of the test particle radial coordinate (dr) measured by observer located at the center of rotation and rotating with angular velocity ω and change of particle radial coordinate (dt_{SO}) measured by stationary observer located at R_{SO} and rotating with angular velocity ω .

$$dt_{SO} = \sqrt{1 - \omega^2 R_{SO}^2} dt, \quad (11)$$

$$dr_{SO} = dr. \quad (12)$$

We should take into account that radial coordinate of stationary observer R_{SO} is constant and does not change in time, while radial coordinate of test particle is time dependent variable i.e.,

$$\frac{dR_{SO}}{dt} = \frac{dR_{SO}}{dt_{SO}} = 0, \quad (13)$$

$$\frac{dr_{SO}}{dt_{SO}} \neq 0, \quad (14)$$

$$\frac{dr}{dt} \neq 0. \quad (15)$$

Velocity of change of test particle radial coordinate o measured by stationary observer could be expressed es follows:

$$(\partial_{t_{SO}} r_{SO})^2 = \frac{1 - \omega^2 r_{SO}^2}{1 - \omega^2 R_{SO}^2} \left(1 - \frac{1 - \omega^2 r_{SO}^2}{E_0^2} \right), \quad (16)$$

when observer is located in the center of the rotation i.e $R_{SO} = 0$ then Eq. 16 transforms to the following expression:

$$(\partial_t r)^2 = (1 - \omega^2 r^2) \left(1 - \frac{1 - \omega^2 r^2}{E_0^2} \right), \quad (17)$$

Eq. 17 coincides with expression derived in [1]

When particle passes stationary observer located at R_{SO} i.e $R_{SO} = r_{SO}$, the Eq. 16 transforms in the following expression

$$(\partial_{t_{SO}} r_{SO})^2 = \left(1 - \frac{1 - \omega^2 R_{SO}^2}{E_0^2} \right), \quad (18)$$

Eq. 18 coincides with expression derived in [8].

From Eq. 16 we could derive expression for test particle radial acceleration measured by stationary observer:

$$\frac{\partial^2 r_{SO}}{\partial t_{SO}^2} = \frac{\omega^2 r_{SO}^2}{1 - \omega^2 R_{SO}^2} \left(1 - \frac{2 (\partial_{t_{SO}} r_{SO})^2 (1 - \omega^2 R_{SO}^2)}{1 - \omega^2 r_{SO}^2} \right). \quad (19)$$

If stationary observer is located in the center of rotation i.e., $R_{SO} = 0$ than Eq. 19 transforms to the following expression:

$$\frac{\partial^2 r}{\partial t^2} = \omega^2 r^2 \left(1 - \frac{2 (\partial_t r)^2}{1 - \omega^2 r^2} \right), \quad (20)$$

Eq. 20 coincides with expression of acceleration derived in [1].

When particle passes stationary observer located at R_{SO} i.e, $R_{SO} = r_{SO}$, the particle radial acceleration measured by stationary observer is expressed as follows:

$$\frac{\partial^2 r_{SO}}{\partial t_{SO}^2} = \frac{\omega^2 R_{SO}^2}{1 - \omega^2 R_{SO}^2} (1 - 2(\partial_{t_{SO}} r_{SO})^2). \quad (21)$$

From Eq. 21 we see that if particle passes observer with radial velocity larger then $1/\sqrt{2}$ i.e, $\partial_{t_{SO}} r_{SO} > 1/\sqrt{2}$, then radial acceleration of the test particle measured by stationary observer is negative, i.e., $\partial^2 r_{SO}/\partial t_{SO}^2 < 0$. From Eq. 19 follows that when particle is located in the center of rotation its radial acceleration measured by stationary observer is negative in the case when $2(\partial_{t_{SO}} r_{SO})^2(1 - \omega^2 R_{SO}^2) < 1$ i.e., velocity measured by stationary observer satisfies condition:

$$(\partial_{t_{SO}} r_{SO})^2(1 - \omega^2 R_{SO}^2) > 1/2. \quad (22)$$

After taking into account transformations Eq. 11 and Eq. 12 we can see that condition given by Eq. 22 is satisfied the particle velocity measured by observer located in the center of rotation satisfies following condition:

$$\partial_t r > \frac{1}{\sqrt{2}}. \quad (23)$$

III. CONCLUSIONS

In this study the dynamics of centrifugally accelerated relativistic particle is studied in the frame rigidly rotating observers located at certain distance from the center of rotation. The expressions for the particle radial velocity and radial acceleration are derived. It was analyzed whether any observer could detect reversal of the centrifugal acceleration.

From Eq. 19, Eq. 22 and Eq. 23 follows that radial acceleration of the test particle located in the center of the rotation measured by any stationary observer is negative when particle radial velocity measured by observer located in the center of rotation is larger than $1/\sqrt{2}$. From Eq. 21 follows that particle radial acceleration measured by stationary observer *at the moment when particle passes observer* is negative when at this moment particle radial velocity measured by stationary observer is larger than $1/\sqrt{2}$. These results are in agreement with result obtained in [1]. We could conclude that centrifugal acceleration reversal described

in [1] occurs in the frame of any stationary observer. *When one needs to take derivative of expression in respect of time, one should take into account that radial coordinate of moving particle and observer coincide at the particular moment of time and they are not equal for arbitrary moment of time, in other word one should keep in mind that radial coordinate of stationary observer does not depend on time while the radial coordinate of particle varies in time.*

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